

Heat conduction in a composite sphere - the effect of fractional derivative order on temperature distribution

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Abstract. The aim of the contribution is an analysis of time-fractional heat conduction in a sphere with an inner heat source. The object of the consideration is a solid sphere with a spherical layer. The heat conduction in the solid sphere and spherical layer is governed by fractional heat conduction equation with a Caputo time-derivative. Mathematical (classical) or physical formulations of the Robin boundary condition and the perfect contact of the solid sphere and spherical layer is assumed. The boundary condition and the heat flux continuity condition at the interface are expressed by the Riemann-Liouville derivative. An exact solution of the problem under mathematical conditions is determined. A solution of the problem under physical boundary and continuity conditions using the Laplace transform method has been obtained. The inverse of the Laplace transform by using the Talbot method are numerically determined. Numerical results show the effect of the order of the Caputo and the Riemann-Liouville derivatives on the temperature distribution in the sphere.

Keywords: fractional heat conduction, heating source, Robin boundary condition

1 Introduction

The fundamental of the classical heat transfer theory is the Fourier law which leads to the parabolic partial differential equation of the heat conduction [1]. A consequence of the Fourier's law is unrealistic speed of heat flow in the medium. This inconvenience can be avoided by a generalization of the Fourier law which leads to a fractional heat conduction equation [2]. The heat conduction governed by the fractional differential equation is the subject of papers [3-9]. Applications of fractional order calculus are presented in books [10-12] and papers [13-15].

If the heat transfer in a bounded medium is considered then the heat equation is complemented by boundary conditions. The Dirichlet, Neumann and Robin boundary conditions are often used in describing the heat transfer between the body and the surroundings. In the classical heat theory, the Neumann and Robin boundary conditions include the normal derivative at the boundary of the considered region. Introducing the time-fractional derivative in the Neumann and Robin boundary conditions, the physical formulation of these conditions is obtained [16].

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A solution of the linear fractional differential equation under classical boundary conditions can be determined in an analytical form. To solve the fractional equation under physical boundary conditions, the Laplace technique can be applied. This approach leads to a solution of the problem in the Laplace domain. The temperature distribution in the time domain by using an algorithm for numerical inversion of the Laplace transform can be obtained. The methods of numerical inversion of the Laplace transforms used in classical analysis can be also applied to Laplace transforms obtained by solving the problems with fractional derivatives. Selected methods of numerical inversion of the Laplace transforms are presented in papers [17-20].

In this paper, we present the solution of the fractional heat conduction problem in a sphere consisting of an inner solid sphere and a spherical layer. The mathematical and physical formulation of the Robin boundary conditions is considered. The perfect thermal contact of the inner sphere and the spherical layer is assumed. The effect of the fractional order on the temperature distribution in the sphere has been numerically investigated.

2 Formulation of the problem

We consider the time-fractional radial heat conduction problem in a sphere. The two regions of the sphere are distinguished: $0 \leq r \leq r_1$ - a solid inner sphere and $r_1 \leq r \leq b$ - a spherical layer, where r is the radial coordinate. The heat transfer in the regions is governed by the fractional heat conduction equation [3]:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T_i}{\partial r} \right) + \frac{1}{\lambda_i} g_i(r, t) = \frac{1}{a_i} \frac{\partial^{\alpha_i} T_i}{\partial t^{\alpha_i}}, \quad i = 1, 2 \quad (1)$$

where $g_i(r, t)$ is the volumetric rate of heat generation, a_i is the thermal diffusivity, λ_i is the thermal conductivity and α_i denotes the fractional order of the Caputo derivative with respect to time t . The Caputo derivative is defined by [21]

$${}_0^C D_t^\alpha f(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha \leq m \\ f^{(m)}(t), & \alpha = m \in N \end{cases} \quad (2)$$

where Γ denotes the gamma function. The boundary condition and the continuity condition at interface are assumed in a form with the Riemann-Liouville fractional derivative $D_{RL}^{1-\beta}$ which is defined by [21]

$$D_{RL}^\beta f(t) = \frac{d}{dt} \left[\frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f(\tau)}{(t-\tau)^\beta} d\tau \right], \quad 0 < \beta \leq 1 \quad (3)$$

On the outer surface of the sphere, the Robin boundary condition [16] is assumed

$$\lambda_2 D_{RL}^{1-\beta_2} \frac{\partial T_2}{\partial r}(b, t) = a_\infty (T_\infty(t) - T_2(b, t)) \quad (4)$$

where a_∞ is the outer heat transfer coefficient and T_∞ is the ambient temperature. The perfect thermal contact at the interface between the inner sphere and the spherical layer is described by conditions:

$$T_1(r_1, t) = T_2(r_1, t) \quad (5)$$

$$\lambda_1 D_{RL}^{1-\beta_1} \frac{\partial T_1}{\partial r}(r_1, t) = \lambda_2 D_{RL}^{1-\beta_2} \frac{\partial T_2}{\partial r}(r_1, t) \quad (6)$$

and the initial condition is

$$T(r, 0) = F_i(r) . \quad (7)$$

The conditions (4) and (6) for $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$ are called the physical conditions [16], if $\beta_1 = \beta_2 = 1$, there are so called mathematical conditions. In the second case, the D_{RL}^0 in equations (4) and (6) means an identity operator and can be omitted. We further consider the case of $\alpha_i \in (0, 1]$.

3 Solution to the problem

In order to transform the heat conduction equation (1) into the fractional equation with a constant coefficient, we introduce new functions $U_i(r, t)$ given by the following relationship

$$U_i(r, t) = r(T_i(r, t) - T_\infty(t)), \quad i = 1, 2 \quad (8)$$

Taking into account equation (8) in the initial-boundary problem (1) and (4-7), we obtain formulation of the problem for the function $U_i(r, t)$ in the form

$$a_i \frac{\partial^2 U_i(r, t)}{\partial r^2} + r g_i^*(r, t) = \frac{\partial^\alpha U_i(r, t)}{\partial t^{\alpha_i}}, \quad i = 1, 2 \quad (9)$$

$$\lambda_2 D_{RL}^{1-\beta_2} \left(\frac{\partial U_2}{\partial r}(b, t) - \frac{1}{b} U_2(b, t) \right) = -a_\infty U_2(b, t) \quad (10)$$

$$U_1(r_1, t) = U_2(r_1, t) \quad (11)$$

$$\lambda_1 D_{RL}^{1-\beta_1} \left(\frac{\partial U_1}{\partial r}(r_1, t) - \frac{1}{r_1} U_1(r_1, t) \right) = \lambda_2 D_{RL}^{1-\beta_2} \left(\frac{\partial U_2}{\partial r}(r_1, t) - \frac{1}{r_1} U_2(r_1, t) \right) \quad (12)$$

$$U_i(r, 0) = r(F_i(r) - T_\infty(0)), \quad i = 1, 2 \quad (13)$$

Moreover, the conditions (10-12) are complemented by a condition for $r = 0$, which is obtained using equation (8)

$$U_1(0,t) = 0 \quad (14)$$

The functions $g_i^*(r,t)$ in equation (9) are given by the formula

$$g_i^*(r,t) = \frac{a_i}{\lambda_i} g_i(r,t) - \frac{d^{\alpha_i} T_\infty(t)}{dt^{\alpha_i}} \quad (15)$$

The solutions of the initial-boundary problem (9-14) for mathematical and physical formulations will be presented below.

3.1 Mathematical formulation of boundary and continuity conditions

The heat conduction problem (9-14) under mathematical conditions for $\alpha_1 = \alpha_2 = \alpha$ can be solved analytically. We search for the solution to this problem in the form of the series of orthogonal functions $\Psi_{i,k}(r)$:

$$U_i(r,t) = \sum_{k=1}^{\infty} \Lambda_k(t) \Psi_{i,k}(r), \quad i = 1, 2 \quad (16)$$

In the first step, we find the functions $\Psi_{i,k}(r)$ by solving the following eigenvalue problem

$$\frac{d^2 \Psi_{i,k}(r)}{dr^2} + \frac{\gamma_k^2}{a_i} \Psi_{i,k}(r) = 0 \quad i = 1, 2 \quad (17)$$

$$\Psi_{1,k}(0) = 0 \quad (18)$$

$$\Psi_{1,k}(r_1) = \Psi_{2,k}(r_1) \quad (19)$$

$$\lambda_1 \left(\frac{d\Psi_1(r_1)}{dr} - \frac{1}{r_1} \Psi_1(r_1) \right) = \lambda_2 \left(\frac{d\Psi_2(r_1)}{dr} - \frac{1}{r_1} \Psi_2(r_1) \right) \quad (20)$$

$$\frac{d\Psi_2(b)}{dr} = \left(\frac{1}{b} - \frac{a_\infty}{\lambda_2} \right) \Psi_2(b) \quad (21)$$

The functions $\Psi_{i,k}(r)$ are

$$\Psi_{1,k}(r) = B_{1,k} \sin \mu_{1,k} r \quad (22)$$

$$\Psi_{2,k}(r) = A_{2,k} \cos \mu_{2,k}(r - r_1) + B_{2,k} \sin \mu_{2,k}(r - r_1) \quad (23)$$

where $\mu_{i,k} = \gamma_k / \sqrt{a_i}$ and γ_k are roots of the equation

$$Q_1 \lambda_2 \mu_1 \sin \mu_1 r_1 + Q_2 Q_3 = 0 \quad (24)$$

where

$$Q_1 = \left(\frac{a_\infty}{\lambda_2} - \frac{1}{b} \right) \cos \mu_2 (b - r_1) - \mu_2 \sin \mu_2 (b - r_1),$$

$$Q_2 = \left(\frac{a_\infty}{\lambda_2} - \frac{1}{b} \right) \sin \mu_2 (b - r_1) + \mu_2 \cos \mu_2 (b - r_1)$$

and

$$Q_3 = \lambda_1 \mu_1 \cos \mu_1 r_1 + \frac{\lambda_2 - \lambda_1}{r_1} \sin \mu_1 r_1.$$

These functions fulfil the orthogonality condition in the form

$$\frac{\lambda_1}{a_1} \int_0^{r_1} \Psi_{1,k}(r) \Psi_{1,k'}(r) dr + \frac{\lambda_2}{a_2} \int_{r_1}^b \Psi_{2,k}(r) \Psi_{2,k'}(r) dr = \begin{cases} 0 & \text{for } k' \neq k \\ N_k & \text{for } k' = k \end{cases} \quad (25)$$

The coefficients $B_{1,k}$, $A_{2,k}$, $B_{2,k}$, occurring in equations (22-23), are determined by using conditions (19-21). Assuming $B_{1,k} = 1$, we obtain $A_{2,k} = \sin \mu_{1,k} r_1$ and $B_{2,k} = Q_3 / \lambda_2 \mu_{2,k}$.

The function $\Lambda_k(t)$ satisfies a fractional equation which is obtained substituting equation (16) into equation (9) and using orthogonality condition (25). The initial condition for the function $\Lambda_k(t)$ is obtained in a similar way by using (16) and (25) in condition (13). The fractional differential equation and initial condition have the form

$$\frac{d^\alpha \Lambda_k(t)}{dt^\alpha} + \gamma_k^2 \Lambda_k(t) = \frac{1}{N_k^r} \left(\int_0^{r_1} r g_1^*(r, t) \Psi_{1,k}(r) dr + \int_{r_1}^b r g_2^*(r, t) \Psi_{2,k}(r) dr \right) \quad (26)$$

$$\Lambda_k(0) = \frac{1}{N_k^r} \left(\frac{\lambda_1}{a_1} \int_0^{r_1} r (F_1(r) - T_\infty(0)) \Psi_{1,k}(r) dr + \frac{\lambda_2}{a_2} \int_{r_1}^b r (F_2(r) - T_\infty(0)) \Psi_{2,k}(r) dr \right) \quad (27)$$

where $N_k^r = \frac{\lambda_1}{a_1} \int_0^{r_1} [\Psi_{1,k}(r)]^2 dr + \frac{\lambda_2}{a_2} \int_{r_1}^b [\Psi_{2,k}(r)]^2 dr$. A solution of the problem (26-27)

is given by [22]

$$\Lambda_k(t) = \frac{1}{N_k^r} \left(\int_0^{r_1} r \Psi_{1,k}(r) \int_0^t (t-\tau)^{\alpha-1} g_1^*(r, \tau) E_{\alpha, \alpha}(-\gamma_k^2 (t-\tau)^\alpha) d\tau dr \right. \\ \left. + \int_{r_1}^b r \Psi_{2,k}(r) \int_0^t (t-\tau)^{\alpha-1} g_2^*(r, \tau) E_{\alpha, \alpha}(-\gamma_k^2 (t-\tau)^\alpha) d\tau dr \right) \quad (28)$$

$$+ \frac{1}{N_k^r} E_{\alpha, 1}(-\gamma_k^2 t^\alpha) \left(\frac{\lambda_1}{a_1} \int_0^{r_1} r (F_1(r) - T_\infty(0)) \Psi_{1,k}(r) dr + \frac{\lambda_2}{a_2} \int_{r_1}^b r (F_2(r) - T_\infty(0)) \Psi_{2,k}(r) dr \right)$$

where $E_\alpha(z)$ is the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \quad (29)$$

Finally, taking into account equation (8) and (16), we obtain the temperature distribution in the sphere under mathematical formulation of the boundary and continuity conditions in the form

$$T_i(r, t) = T_\infty(t) + \frac{1}{r} \sum_{k=1}^{\infty} \Lambda_k(t) \Psi_{i,k}(r), \quad i = 1, 2 \quad (30)$$

where functions $\Lambda_k(t)$ are given by (28) and $\Psi_{i,k}(r)$ are defined by (22) and (23).

If $F_i(r) = T_{init} = const$, $T_\infty(t) = T_\infty = const$ and $g_1(r, t) = G = const$, $g_2(r, t) = 0$, then the functions $\Lambda_k(t)$ are given by

$$\begin{aligned} \Lambda_k(t) = & \frac{G}{N_k^r \mu_{1,k}^2 \gamma_k^2} \frac{a_1}{\lambda_1} (\sin \mu_{1,k} r_1 - \mu_{1,k} r_1 \cos \mu_{1,k} r_1) (1 - E_\alpha(-\gamma_k^2 t^\alpha)) \\ & + \frac{T_{init} - T_\infty}{N_k^r} E_\alpha(-\gamma_k^2 t^\alpha) \left\{ \frac{\lambda_1}{\mu_{1,k}^2 a_1} (\sin \mu_{1,k} r_1 - \mu_{1,k} r_1 \cos \mu_{1,k} r_1) + \frac{\lambda_2}{a_2} \frac{1}{\mu_{2,k}^2} \cdot \right. \\ & \left[A_{2,k} (b \mu_{2,k} \sin \mu_{2,k} (b - r_1) + \cos \mu_{2,k} (b - r_1) - 1) \right. \\ & \left. \left. + B_{2,k} (-b \mu_{2,k} \cos \mu_{2,k} (b - r_1) + \sin \mu_{2,k} (b - r_1) + r_1 \mu_{2,k}) \right] \right\} \end{aligned} \quad (31)$$

3.2 Physical formulation of boundary and continuity conditions

A solution of the heat conduction problem (9-14) under physical boundary and continuity conditions ($\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ in equations (10) and (12)) will be obtained by using the Laplace technique. The Laplace transform is defined as

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt \quad (32)$$

where $f(t)$ for $t > 0$ is a given function of the exponential type and s is a complex parameter. After applying the Laplace transformation to the equations (9-12) and (14), and using the properties of the Laplace transform, we obtain

$$\frac{d^2 \bar{U}_i}{dr^2} - \frac{s^{\alpha_i}}{a_i} \bar{U}_i = \bar{h}_i(r, s) \quad (33)$$

$$\bar{U}_1(0, s) = 0 \quad (34)$$

$$\bar{U}_1(r_1, s) = \bar{U}_2(r_1, s) \quad (35)$$

$$\lambda_1 s^{1-\beta_1} \left(\frac{d\bar{U}_1(r_1, s)}{dr} - \frac{1}{r_1} \bar{U}_1(r_1, s) \right) = \lambda_2 s^{1-\beta_2} \left(\frac{d\bar{U}_2(r_1, s)}{dr} - \frac{1}{r_1} \bar{U}_2(r_1, s) \right) \quad (36)$$

$$\frac{d\bar{U}_2(b,s)}{dr} + \left(\frac{a_\infty}{\lambda_2 s^{1-\beta_2}} - \frac{1}{b} \right) \bar{U}_2(b,s) = 0 \quad (37)$$

where

$$\bar{h}_i(r,s) = r \frac{s^{\alpha_i-1}}{a_i} (s\bar{T}_\infty(s) - F_i(r)) - \frac{r}{\lambda_i} \bar{g}_i(r,s).$$

The solution of the equation (33) for $i = 1$, using the condition (34), can be written in the form

$$\bar{U}_1(r,s) = B_1 \sinh S_1 r + \frac{1}{S_1} \int_0^r \bar{h}_1(u,s) \sinh S_1 (r-u) du \quad (38)$$

and the general solution of equation (33) for $i = 2$, is as follows

$$\bar{U}_2(r,s) = A_2 \cosh S_2 (r-r_1) + B_2 \sinh S_2 (r-r_1) + \frac{1}{S_2} \int_{r_1}^r \bar{h}_2(u,s) \sinh S_2 (r-u) du \quad (39)$$

where $S_i = \frac{s^{\alpha_i/2}}{\sqrt{a_i}}$. Using conditions (35-37), a system of linear equations with respect to unknown constants B_1 , A_2 and B_2 is obtained

$$B_1 \sinh S_1 r_1 - A_2 = P_1 \quad (40)$$

$$B_1 s^{1-\beta_1} \lambda_1 \left(S_1 \cosh S_1 r_1 - \frac{1}{r_1} \sinh S_1 r_1 \right) + A_2 \frac{s^{1-\beta_2} \lambda_2}{r_1} - B_2 s^{1-\beta_2} \lambda_2 S_2 = P_2 \quad (41)$$

$$\begin{aligned} & A_2 \frac{1}{s^{\beta_2}} (a_\infty b s^{\beta_2} S_2 \cosh S_2 (b-r_1) - s S_2 \lambda_2 \cosh S_2 (b-r_1) + b s S_2^2 \lambda_2 \sinh S_2 (b-r_1)) \\ & + B_2 \frac{1}{s^{\beta_2}} (b s S_2^2 \lambda_2 \cosh S_2 (b-r_1) + a_\infty b s^{\beta_2} S_2 \sinh S_2 (b-r_1) - s S_2 \lambda_2 \sinh S_2 (b-r_1)) = P_3 \end{aligned} \quad (42)$$

where

$$\begin{aligned} P_1 &= -\frac{1}{S_1} \int_0^{r_1} \bar{h}_1(u,s) \sinh S_1 (r_1-u) du, \\ P_2 &= -\frac{s^{1-\beta_1} \lambda_1}{S_1 r_1} \left(S_1 r_1 \int_0^{r_1} \bar{h}_1(u,s) \cosh S_1 (r_1-u) du - \int_0^{r_1} \bar{h}_1(u,s) \sinh S_1 (r_1-u) du \right), \\ P_3 &= -\frac{1}{s^{\beta_2}} \left(s \lambda_2 S_2 b \int_{r_1}^b \bar{h}_2(u,s) \cosh S_2 (b-u) du + (a_\infty b s^{\beta_2} - s \lambda_2) \int_{r_1}^b \bar{h}_2(u,s) \sinh S_2 (b-u) du \right). \end{aligned}$$

Substituting the determined constants B_1 , A_2 and B_2 into equations (38-39), we get the complete solution of the problem in the Laplace domain. The temperature distribution in the sphere is given by

$$T_i(r,t) = T_\infty(t) + \frac{1}{r} L^{-1} [\bar{U}_i(r,s)], \quad i=1,2 \quad (43)$$

The inverse of the Laplace transform of the functions $\bar{U}_i(r,s), i=1,2$ is calculated numerically. The fixed Talbot algorithm to numerical inversion of the Laplace transform has been used, [18-19]. Applying this algorithm, the approximate values of the function $U_i(r,t) = L^{-1} [\bar{U}_i(r,s)]$ are determined using the formula

$$U_i(r,t) = \frac{p}{M} \left\{ \frac{1}{2} \bar{U}_i(r,p) \exp(pt) + \sum_{k=1}^{M-1} \operatorname{Re} \left[\exp(ts(\theta_k)) \bar{U}_i(r,s(\theta_k)) (1 + j\sigma(\theta_k)) \right] \right\} \quad (44)$$

where $s(\theta) = p\theta(\operatorname{ctg}\theta + j)$, $\sigma(\theta) = \theta + (\theta \operatorname{ctg}\theta - 1) \operatorname{ctg}\theta$, $p = 2M/(5t)$, $j = \sqrt{-1}$, and $\theta_k = k\pi/M$, M is a number of precision decimal digits.

4 Results of numerical calculations

The effect of the fractional order of time-derivatives occurring in the heat conduction equation and boundary conditions on the temperature distribution in the sphere has been numerically investigated. For the heat conduction model with the mathematical formulation of the boundary and continuity conditions, the results obtained by using the numerical inversion of the Laplace transform has been compared with the exact solution. The calculation has been performed for the following geometrical and thermal data: outer radius of the sphere is $b = 1.0$ [m], the radius of the inner sphere is $\hat{r}_1 = r_1/b = 0.6$, the thermal diffusivities are $a_1 = 3.352 \cdot 10^{-6}$ [m²/s ^{α}], $a_2 = 1.474 \cdot 10^{-5}$ [m²/s ^{α}], the thermal conductivities are $\lambda_1 = 16$ [W/(m·K)], $\lambda_2 = 54$ [W/(m·K)], the outer heat transfer coefficient is $a_\infty = 500$ [W/(m²·K)], the ambient temperature is $T_\infty = 25$ [°C] and the initial temperature is assumed as $T_{init} = 25$ [°C].

Table 1. The non-dimensional temperature $\hat{T}(\hat{r}, \hat{t})$ for $\hat{t} = tb^2/a_2 = 1.0$, calculated by using the exact solution and numerical inversion of the Laplace transform (NILT)

\hat{r}	$\alpha = 0.7$		$\alpha = 0.8$		$\alpha = 0.9$		$\alpha = 1.0$	
	Exact	NILT	Exact	NILT	Exact	NILT	Exact	NILT
0	1.44421	1.44404	2.25696	2.25679	3.73501	3.73484	4.77135	4.77118
0.2	1.44176	1.44175	2.20899	2.20898	3.53158	3.53158	4.44462	4.44462
0.4	1.40995	1.40995	1.98699	1.98699	2.87100	2.87100	3.46198	3.46198
0.6	1.13143	1.13143	1.32024	1.32023	1.61769	1.61768	1.81594	1.81594
0.8	1.03183	1.03182	1.11964	1.11962	1.27291	1.27289	1.37645	1.37643
1.0	1.00653	1.00652	1.03246	1.03245	1.08066	1.08065	1.11348	1.11347

The non-dimensional temperature $\hat{T} = T/T_{init}$ in the selected points of the sphere for the mathematical boundary and continuity conditions ($\beta_1 = \beta_2 = 1.0$, $\alpha_1 = \alpha_2 = \alpha$) for different orders of the Caputo derivative α are presented in Table 1. The volumetric rate of heat generation in the inner sphere is $g_1 = 20000[W/m^3]$ and in the spherical layer $g_2 = 0$. The results presented in the Table 1 fulfil the condition: $|Exact - NILT|/Exact < 3.6 \cdot 10^{-5}$. The small differences of the Exact and NILT results allow for the use of the algorithm of numerical inversion of the Laplace transform to the heat conduction problem under physical formulation of boundary and continuity conditions.

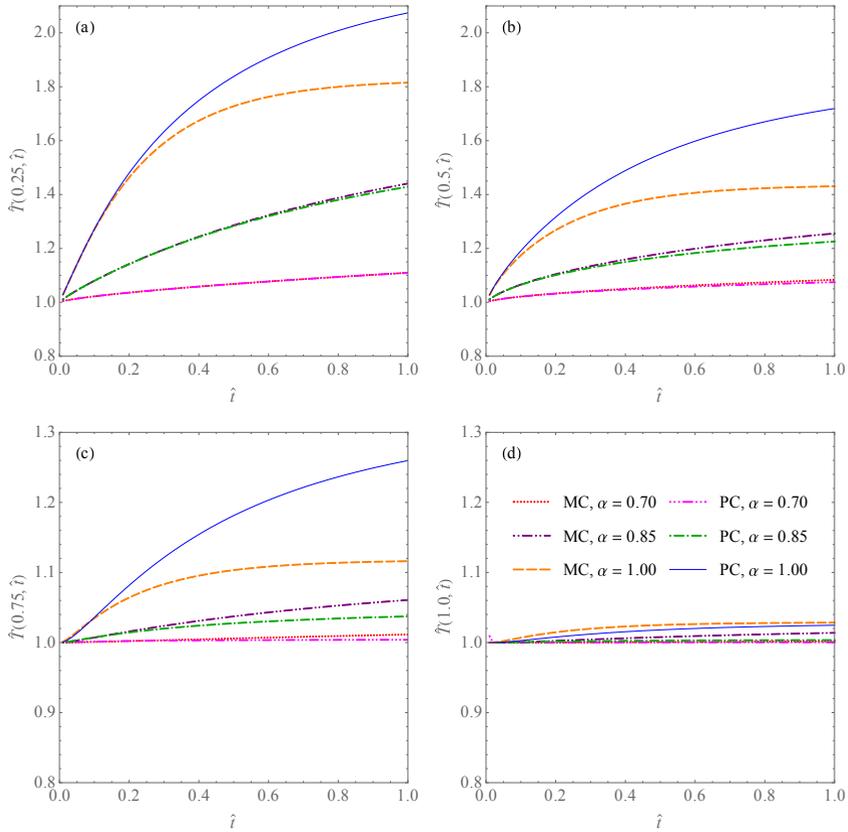


Fig. 1. The non-dimensional temperature $\hat{T}(\hat{r}, \hat{t})$ as a function of time \hat{t} for various values of fractional derivatives α and β : (a) $\hat{r} = 0.25$; (b) $\hat{r} = 0.5$; (c) $\hat{r} = 0.75$; (d) $\hat{r} = 1.0$

The time-histories temperatures in selected points of the sphere are presented in Fig. 1. The calculations were performed for different orders of the derivative occurring in the heat equation. In the problem under mathematical conditions (MC) one was assumed $\alpha_1 = \alpha_2 = \alpha = 0.7; 0.85; 1.0$ and $\beta_1 = \beta_2 = 1.0$ and for the problem under physical conditions (PC) the calculations were performed for $\alpha_1 = \alpha = 0.7; 0.85; 1.0$, $\alpha_2 = \beta_2 = 0.9$ and $\beta_1 = \alpha_1$. The volumetric rate of heat generation was assumed: $g_1 = 5000[W/m^3]$ and $g_2 = 0$. As was expected, the differences between the temperatures obtained for various orders of α 's decrease if the distance from the source increases. The significant effect on

the temperature distribution in the sphere has order of the derivatives in the heat conduction model.

Conclusions

The problem of the fractional heat conduction in a sphere with inner heat source under mathematical and physical Robin boundary and continuity conditions by using the Laplace transform technique has been solved. It was noted a high agreement of the results obtained on the basis of the exact solution and the results computed by using numerical inversion of the Laplace transform for the heat conduction under mathematical conditions. The effect of the order of the Caputo derivative occurring in the heat conduction equation on the temperature distribution in the sphere has been numerically investigated. It was stated that the temperatures in the sphere obtained for models with the same fractional order α with mathematical and physical Robin conditions differ slightly. The significant differences in the temperatures have been observed for different orders of the fractional derivative in the heat conduction equation.

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