Prediction of modal properties of circular disc with pre-stressed fields

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Abstract. Many structural elements in technical devices, as well as some tools in manufacturing equipment, such as saw blade or , have a geometric shape similar to a circular disc. In many cases, these circular discs must meet the required dynamic properties. One of the techniques to achieve the required dynamic properties of circular disc is based on initiating pre-stressed fields in the disc plane. In-plane residual stresses are created by appropriate technological treatments in selected disc part, for example by rolling of annulus leading to plastic deformation which causes the change of volume distribution in this disc part. The effects of in-plane residual stresses on modal properties of circular disc are analysed. The natural frequencies modifications depending on position and width and also on the change in thickness of the roll-prestressed annulus are investigated in this paper.

Keywords: circular disc, vibration, modal properties, pre-stressed fields, finite element method

1 Introduction

There are many situations and applications in the acoustical and structural problems where the vibrating circular disc have to be modified to achieve convenient dynamical properties [1]. This is mainly concerned with the requirements for the so-called "tuning" of the modal properties of the vibrating circular disc by means of technological treatments that induce the residual stresses in the disc plane. The natural frequencies of circular disc, clamped on inner radius, are varying when the localized plastic deformation caused for example by roll-tensioning, induces residual stresses. The natural frequencies of circular disc, clamped on inner radius, are varying when the localized plastic deformation caused for example by roll-tensioning, induces residual stresses. A similar effect resulting from residual stresses can also be achieved by phase transformation during technological treatments. During the process of initiating disc in-plane stresses using the roll-tensioning process, a disc is compressed within a certain annular contact zone between two opposing rollers. The contact zone of circular disc is plastically deformed and the residual stresses are occurred in whole disc plane. Then the effects of residual stresses induced by roll-tension on modal properties (natural frequencies, mode shapes) can be analysed. The appropriate conditions for corresponding technological treatments can be predicted using the natural frequency

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characteristics where the natural frequency values are depending on parameters causing in-plane disc stresses. The considered computational procedure for the implementation of in-plane disc residual stresses is based on idea which is similar to the formation of thermoelastic stresses.The natural frequency characteristics for various rolling positions, for various rolling depths and widths of the annulus are obtained by modal analysis using Finite Element Method (FEM). The role of residual stresses obtained by rolling can be assessed from the change in natural frequencies and modal shapes.

2 Formulation of the problem

Creating pre-stressed fields in a circular disk causes a change in its spatial properties, i.e. a change in the distribution of its mass and stiffness parameters. As a consequence of these changes, the modification of the dynamic properties of a given circular disc occurs.

2.1 Theoretical approach to modelling of a modified dynamic system

The general equation of motion for undamped system without external excitation is defined by

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{0}\,, \tag{1}$$

where $\mathbf{M}$ is mass matrix, $\mathbf{K}$ is stiffness matrix of the system, $\mathbf{u}$ and $\ddot{\mathbf{u}}$ are displacements and accelerations vectors, respectively.

Equation (1) can be transformed [5] using the transformation equations

$$\mathbf{u}(t) = \mathbf{\Phi}\mathbf{q}(t)\,, \qquad \mathbf{\Phi}^T\mathbf{M}\mathbf{\Phi} = \mathbf{I}\,, \qquad \mathbf{\Phi}^T\mathbf{K}\mathbf{\Phi} = \mathbf{\Lambda} \tag{2}$$

where $\mathbf{\Phi} = [\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_N]$ is matrix of modal vectors, $\mathbf{\Lambda}$ is spectral matrix..

Eigenvalue problem of the system (1) can be written in the form

$$(\mathbf{\Phi}^\mathbf{T}\mathbf{K}\mathbf{\Phi} - \omega^2\mathbf{\Phi}^\mathbf{T}\mathbf{M}\mathbf{\Phi})\mathbf{q} = (\mathbf{\Lambda} - \omega^2\mathbf{I})\mathbf{q} = \mathbf{0}\,, \tag{3}$$

where ω is natural angular frequency.

When the system described by equation (1) is modified [4], then the modification of the system has to be incorporated through mass and stiffness changes of system parameters. The mass and stiffness properties of the system are modified and equation (1) becomes

$$(\mathbf{M} + \Delta\mathbf{M})\ddot{\mathbf{u}} + (\mathbf{K} + \Delta\mathbf{K})\mathbf{u} = \mathbf{0}\,, \tag{4}$$

The modification matrices $\Delta\mathbf{M}$ and $\Delta\mathbf{K}$ characterise the mass and stiffness modifications in the spatial model. The practical modification is not carried out on matrices but on physical components or parameters of the structure.

Using equations (2), the eigenvalue problem of modified system (4) is

$$[\mathbf{\Phi}_m^T(\mathbf{K} + \Delta\mathbf{K})\mathbf{\Phi}_m - \omega_m^2\mathbf{\Phi}_m^T(\mathbf{M} + \Delta\mathbf{M})\mathbf{\Phi}_m]\mathbf{q}_m = \mathbf{0}\,, \tag{5}$$

where ω_m is the natural angular frequency of modified system.

Equation (5) provides the new natural angular frequencies (ω_m) and new modal vectors ($\boldsymbol{\phi}_m$) of the system after structural modification. Then i-th natural angular frequency is expressed

$$\omega_i = \sqrt{\frac{\boldsymbol{\phi}_i^T(\mathbf{K} + \Delta\mathbf{K}_\sigma)\boldsymbol{\phi}_i}{\boldsymbol{\phi}_i^T(\mathbf{M} + \Delta\mathbf{M})\boldsymbol{\phi}_i}}\,. \tag{6}$$

2.2 Computational model of circular disc with prestressed fields

The general shape of circular disc of outer radius r_0, inner radius r_v and thickness h_0 (Fig.1) is considered. The material of circular disc is isotropic and homogeneous. The inner radius r_v specifies a circle where the disc is clamped by flanges. To modify the modal properties of a circular disc the case with pre-stressed annulus field is considered. The geometrical shape of disc in-plane prestressed fields are defined by middle radius and width of each prestressed field, i.e. r_1 and b_1.

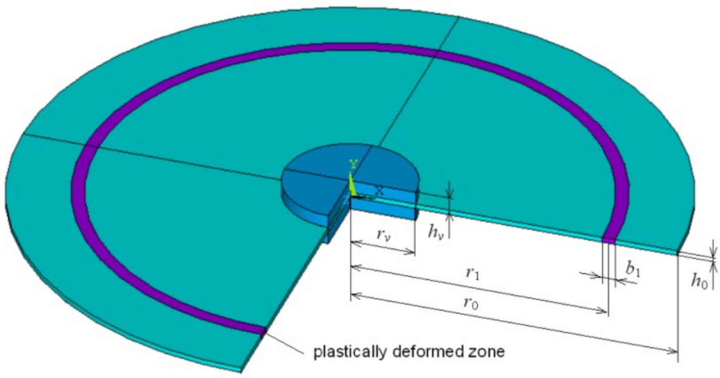

Fig. 1. The circular disc with prestressed fields

The fundamental considerations and derivation of equations of motion are based on Kirchhoff's plate theory assumptions. Using Kirchhoff's plate theory, the field of displacements in the cylindrical coordinates r, φ, z, can by written as

$$\begin{aligned} u_r(r,\varphi,z,t) &= u(r,\varphi) - z\frac{\partial w(r,\varphi,t)}{\partial r}, \\ v_\varphi(r,\varphi,z,t) &= v(r,\varphi) - \frac{z}{r}\frac{\partial w(r,\varphi,t)}{\partial \varphi}, \\ w_z(r,\varphi,z,t) &\cong w(r,\varphi), \end{aligned} \tag{7}$$

where $u(r, \varphi)$, $v(r, \varphi)$ and $w(r, \varphi)$ are displacements of point laying on neutral plane of the circular disc in coordinate directions.

Generally, the strain-displacement relations in the cylindrical coordinates for pre-stressed circular disc can be written in the form

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_0 - \widetilde{H}(r)\boldsymbol{\varepsilon}_i, \tag{8}$$

resp.

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\varphi \\ \gamma_{r\varphi} \end{bmatrix} = \begin{bmatrix} u,_r \\ \dfrac{v,_\varphi}{r} + \dfrac{u}{r} \\ \dfrac{u,_\varphi}{r} + v,_r - \dfrac{v}{r} \end{bmatrix} - z\begin{bmatrix} w,_{rr} \\ \dfrac{w,_{\varphi\varphi}}{r^2} + \dfrac{w,_r}{r} \\ \dfrac{2w,_{r\varphi}}{r} - \dfrac{2w,_\varphi}{r^2} \end{bmatrix} - \widetilde{H}(r)\begin{bmatrix} \varepsilon_{r,i} \\ \varepsilon_{\varphi,i} \\ \gamma_{r\varphi,i} \end{bmatrix} \tag{9}$$

where $\widetilde{H}(r) = H[(r_1 - b_1/2) - r] - H[(r_1 + b_1/2) - r]$ is a modified Heaviside function describing the position of the pre-stressed zone ($H(x) = 0$ for $x < 0$; $H(x) = 1$ for $x > 0$), $\varepsilon_{r,i}$,

$\varepsilon_{\varphi,i}$, $\gamma_{r\varphi,i}$ are the initial strains inserted in pre-stressed area, $\dfrac{\partial(\cdot)}{\partial r} = (\cdot)_{,r}$ and $\dfrac{\partial(\cdot)}{\partial \varphi} = (\cdot)_{,\varphi}$ are the partial derivations.

Generally, the stress-strain relations under consideration of initial stresses and initial strains are given by

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon}_0 - \widetilde{H}(r)\boldsymbol{\varepsilon}_i) + \boldsymbol{\sigma}_i \tag{10}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are stress and strain vectors, $\boldsymbol{\sigma}_i$ and $\boldsymbol{\varepsilon}_i$ are initial stress and initial strain vectors, $\mathbf{D}$ is elasticity matrix. Using the finite element formulation, the equation of motion for a free vibration of in-plane stressed disc is described by expression

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{K} + \Delta\mathbf{K}_\sigma)\mathbf{u} = \mathbf{0}\,, \tag{11}$$

where $\mathbf{M}$ is mass matrix, $\mathbf{K}$ is stiffness matrix, $\mathbf{K}_\sigma$ is stiffness matrix resulting from stress distribution induced by rolling, $\ddot{\mathbf{u}}$ and $\mathbf{u}$ are vector of nodal accelerations and vector of nodal displacements, respectively. We note, that the mass distribution of circular disc after rolling is not changed, but the bending stiffness is considerably changed.

Equation (10) can be transformed to modal coordinates using the transformation equations (2). After applying the these transformations, the equation of motion (11) can be used to determination of the natural angular frequencies and mode shapes of the circular disc with roll-tensioning induced residual stress distribution. We obtain the following eigenvalue problem

$$(\mathbf{K} + \mathbf{K}_\sigma - \omega_i^2\mathbf{M})\boldsymbol{\phi}_i = \mathbf{0}\,, \tag{12}$$

where $\omega_i = \sqrt{\dfrac{\boldsymbol{\phi}_i^T(\mathbf{K} + \mathbf{K}_\sigma)\boldsymbol{\phi}_i}{\boldsymbol{\phi}_i^T\mathbf{M}\boldsymbol{\phi}_i}}$ is i-th natural angular frequency, $\boldsymbol{\phi}_i$ is eigenvector describing i-th modal shape of the circular disc.

3 Numerical simulation and results

We consider a circular disc (Fig. 1) of the outer radius r_0 = 120 mm, flange radius is r_v = 25 mm, thickness h = 1.8 mm. The width of plastically deformed annulus is assumed as 10 mm. This width is selected arbitrarily and it is considered as a representative value for planned experimental verification of investigated phenomenon. The input data used for numerical analysis of circular disc are introduced in the Table 1.

Table 1. Input data

Young modulus	E	**[GPa]**	210.0
Poisson number	μ	**[-]**	0.3
density	ρ	**[kgm^{-3}]**	7800.0
coefficient of thermal expansion	α	**[K^{-1}]**	1.2×10^{-5}
outer radius of disc	r_0	**[mm]**	120.0
flange radius	r_v	**[mm]**	25.0
disc thickness	h_0	**[mm]**	1.8
width of plastic-afected zone	b_1	**[mm]**	10.0
depth of rolling of the plastic-afected zone	Δz	**[mm]**	{0.0; 1.0; 2.0; 3.0; 4.0}

The analysed disc is assumed to be perfectly fixed in region $r \geq r_v$. The outer edge of circular disc is free.

The change in disk stiffness after rolling, which is represented by the modified stiffness matrix $\mathbf{K}_\sigma$, must be determined from the residual stress distribution in the disc plane. To determine the residual stress distribution, the method of thermoelastic stress loading is used [2]. The thermoelastic expansion induces a stress distribution, which is analogous to the stress distribution initiated by rolling. The dependence between temperature and depth of roll-tensioning is approximately described by equation

$$\Delta T \approx \frac{\mu}{h_0 \alpha} \Delta z \,, \tag{13}$$

where μ is Poisson number, α is the coefficient of thermal expansion, h_0 is disc thickness and Δz is depth of roll-tensioning.

The matrices $\mathbf{M}$, $\mathbf{K}$ and additional matrix $\mathbf{K}_\sigma$, which follows from stress distribution arising from rolling (in this model analogy with thermoelastic expansion is used), are calculated automatically by ANSYS. The calculation processes for determination of natural angular frequencies and mode shapes are realised by ANSYS.

The distribution of radial σ_r and tangential σ_t residual stresses induced in plane of circular discs with one roll-tension annulus for various parameters is shown in Fig. 2.

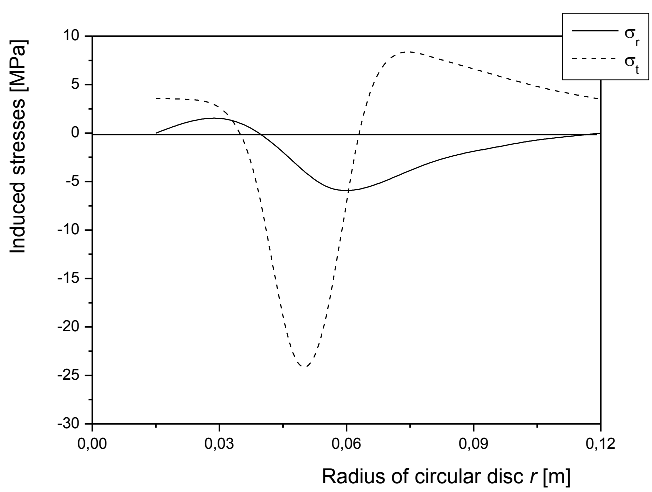

Fig. 2. Distribution of radial σ_r and tangential σ_t residual stresses in disc plane (r_1 = 50 mm; Δz = 1 μm; b_1 = 10 mm)

In Fig. 3 the natural frequency curves of the modal shapes 0/1, 0/0, 0/2, 0/3 (nodal circles/nodal lines) for different depth of rolling (Δz = 1,0 ÷ 4,0 μm) calculated by FEM when r_1 varies from 0,03 m to 0,11 m are shown. The natural frequencies of circular disc before roll-tensioning are marked by r_1 = 0.0 m. The tendency of curves for mode shapes 0/1 and 0/0 differs from curves for mode shapes 0/2 and 0/3. The natural frequencies of the mode shapes 0/2 and 0/3 increase with r_c until the maximum values near $r_c \approx 0.055$ m are reached; then they decrease. Contrary to this, the natural frequencies of the mode shapes 0/1 and 0/0 decrease with r_1 and for $r_1 \approx 0.046$ m reach the minimum; then they increase.

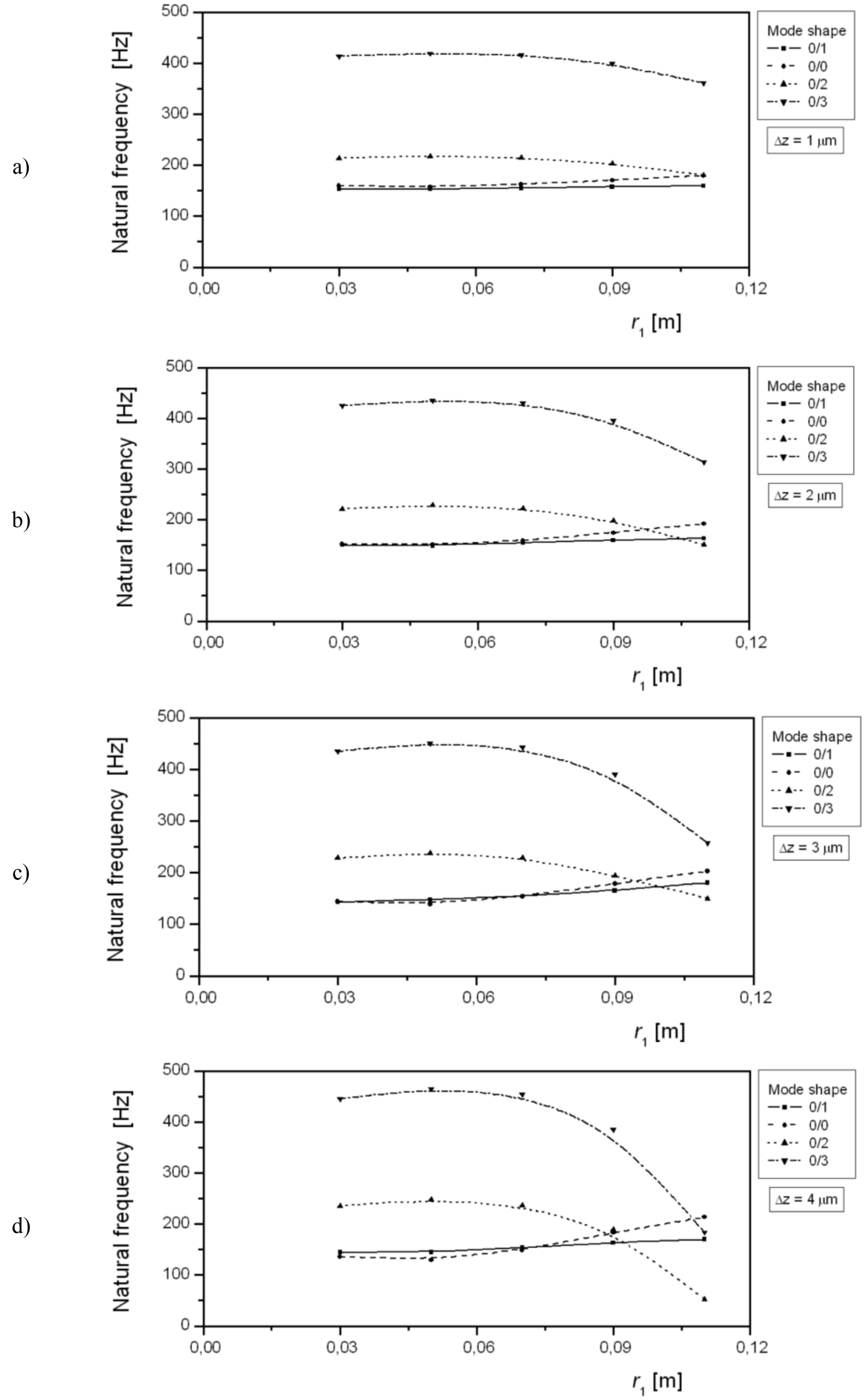

Fig. 3. Dependency of natural frequency on center radius of pre-stressed annulus r_1 for individual Δz and for first four mode shapes

In Fig. 4, the natural frequency curves of the individual modal shapes 0/1, 0/0, 0/2, 0/3 in dependency of depth of rolling are shown. The natural frequencies of the modal shapes 0/1 and 0/0 decrease with r_1 until the minimum values near $r_1 \approx 0.046$ m are reached; then they increase. Contrary to this, the natural frequencies of the modal shapes 0/2 and 0/3 increase with r_c and for $r_1 \approx 0.055$ m reach the minimum; then they increase. The effect of depth of rolling on natural frequency is evident from these graphs.

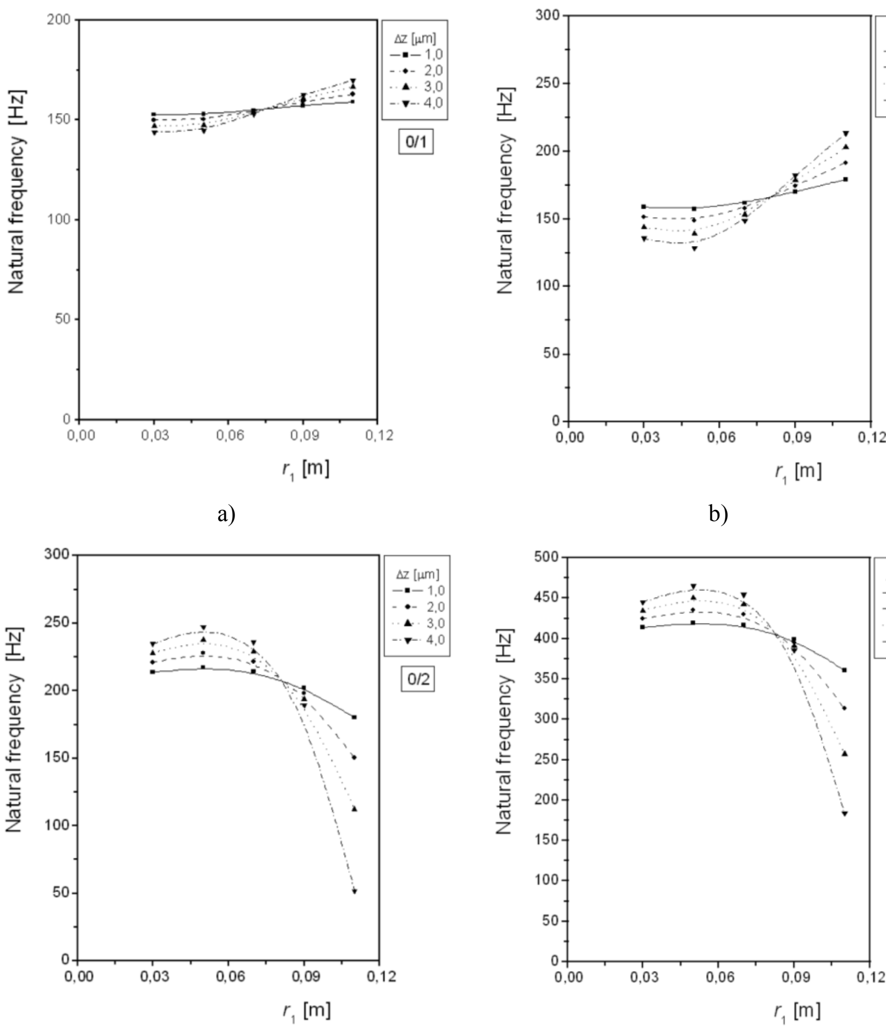

Fig. 4. Dependency of natural frequency of the first four mode shapes on center radius of pre-stressed annulus r_1 and for various Δz

Conclusions

The theoretical formulation and calculation model for analysis of dynamical properties of circular disc with residual stress distribution are presented. Finite element analysis for estimating the natural frequencies was used. For certain mean radius of pre-stressed annulus r_1, the natural frequencies of mode shapes 0/2 and 0/3 become smaller than those before roll-tensioning and the requred effect leading to growth of natural frequencies caused by pre-stresses state of circular disc cannot be achieved. Therefore, the appropriate rolling position is necessary to be determined from natural frequency characteristics calculated for various mean radius of pre-stressed annulus. This method of structural modification can be used to solve the various design problems of mechanical systems [3], [6] and it is very effective to modification of the dynamic properties of similar structural elements.

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References

1. G. Izrael, J. Bukoveczky, L. Gulan, *Influence of Nonstandard Loads onto Life of Chosen Modules of Mobile Working Machines*. Machine Design **3**, 13-16 (2011)
2. M. Sága, R. Bednár, M. Vaško, *Contribution to modal and spectral interval finite element analysis*. Vibration Problems ICOVP 2011, Springer Proceedings in Physics **139**, 269-274 (2011)
3. M. Naď, *Modification of Modal Characteristics of Vibrating Structural Elements*. (Scientific Monographs, Köthen, 2010)
4. F. Kuratani, S. Yano, *Vibration Analysis of a Circular Disc Tensioned by Rolling Using Finite Element Method*. Archive of Applied Mechanics **70**, 279-288 (2000)
5. I. Mazurkievič, L. Gulan, G. Izrael, *Mobile Working Machines - Theory and Constructions of Basic Modules*. [in slovak] (1st ed., STU Bratislava, 2013)
6. M. Orečný, Š. Segľa, R. Huňady, *Methodology for Tuning a Semi-active Dynamic Vibration Absorber on a Passive Suspended Seat*. Applied Mechanics and Materials **81**, 63-68 (2015)